

Subcell Remapping Method on Staggered Polygonal Grids for Arbitrary Lagrangian-Eulerian Methods

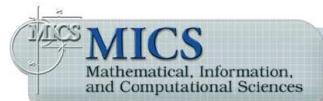
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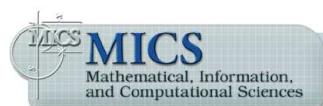
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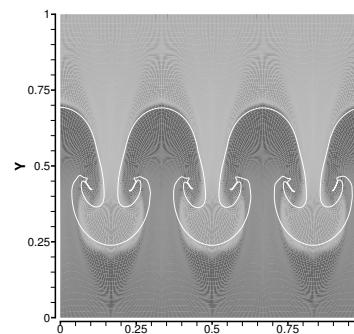
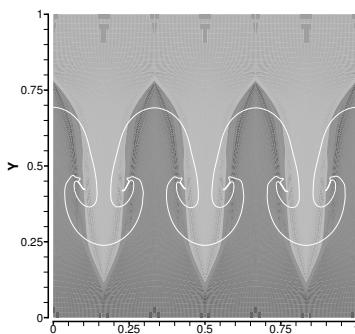
Outline

- **Arbitrary Lagrangian-Eulerian (ALE) Methods**
- **Lagrangian Stage - Discretization, Subcell forces, Artificial Viscosity**
- **Rezone Stage - Reference Jacobian Strategy, Untangling**
- **Staggered Remap**
 - Statement, Requirements and Main Stages
 - Gathering Stage
 - Subcell Remapping Stage
 - Scattering Stage
 - Numerical Examples
- **References**



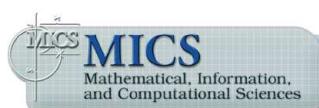
Arbitrary Lagrangian-Eulerian (ALE) Methods

- **Lagrangian methods** — grid is moving with fluid
- **Eulerian methods** — grid is stationary, fluid moving through the grid
- **Arbitrary Lagrangian-Eulerian methods** — grid movement is arbitrary and can be used to improve robustness and accuracy



Rayleigh-Taylor instability problem: Lagrangian - left, ALE - right.

New class of problems can be solved, that cannot be handled by traditional Lagrangian or Eulerian methods



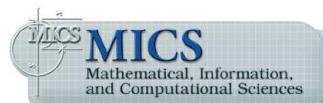
ALE - Main Phases

- **Explicit Lagrangian (solving Lagrangian equations) phase — grid is moving with fluid**
- **Rezone phase — changing the mesh (improving geometrical quality, smoothing, adaptation)**
- **Remap phase (conservative interpolation) — remapping flow parameters from Lagrangian grid to rezoned mesh**

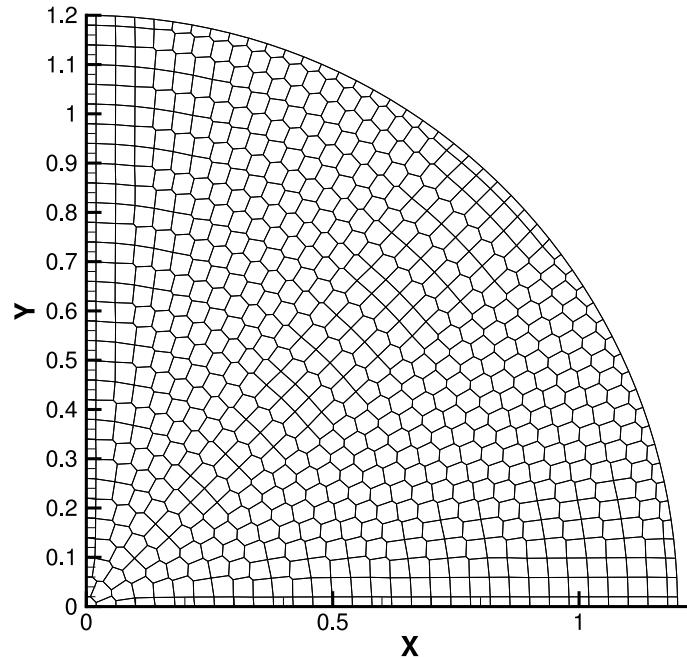


Lagrangian Stage - Discretization

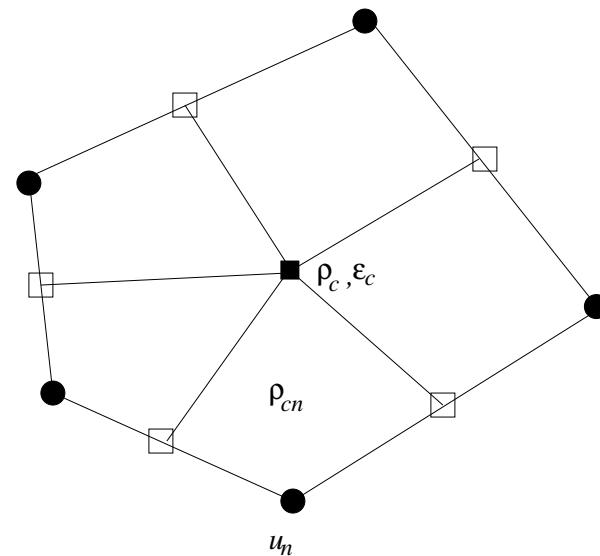
- **Class of meshes: Unstructured (general polygons)**
- Primary variables are defined on a staggered mesh: density centers of cells and centers of the subcells, internal energy - centers of cells, velocity - vertices of cells.
- Conservation - mass, momentum, total energy
- Mimetic artificial viscosity
- Preventing parasitic grid motion



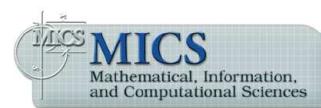
General Polygonal Staggered Mesh



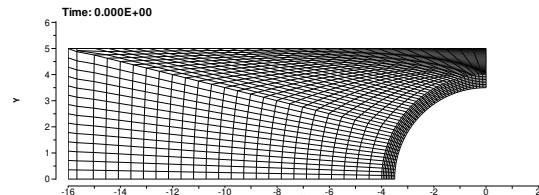
General Polygonal Mesh



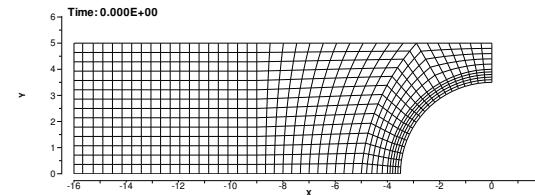
Staggered Discretization



Structured versus Unstructured Grids

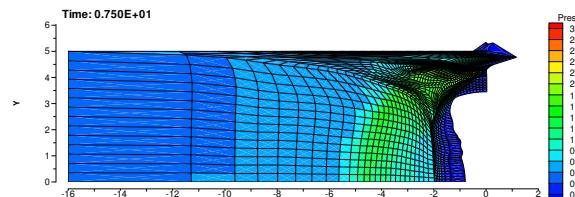


a)

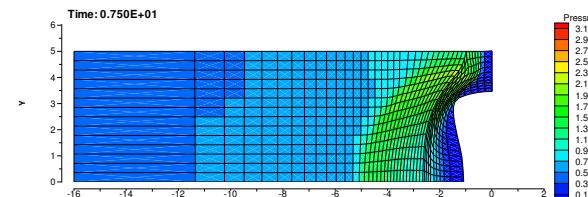


b)

Shape charge problem, Initial Mesh: a) Structured grids, b) Unstructured grid

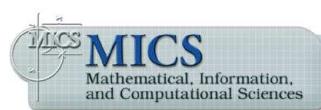


a)



b)

Shape charge problem: a) Structured grids, b) Unstructured grid



Staggered Grid Discretization

Lagrangian subcell masses - m_{cn}

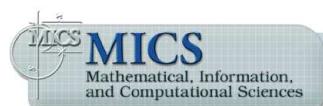
Mass of the cell and of the node

$$m_c \stackrel{def}{=} \sum_{n \in N(c)} m_{cn}, \quad m_n \stackrel{def}{=} \sum_{c \in C(n)} m_{cn}$$

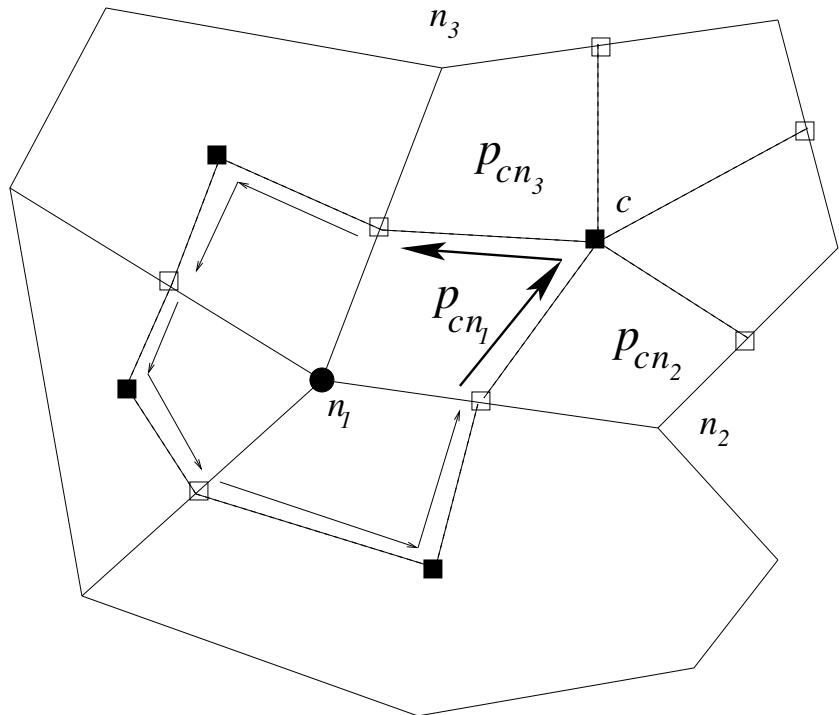
Compatible Discretizations - Utilizing Conservation of Total Energy

$$m_n \frac{d\mathbf{u}_n}{dt} = \sum_{c \in C(n)} \mathbf{f}_c^n, \quad m_c \frac{d\varepsilon}{dt} = - \sum_{n \in N(c)} \mathbf{f}_c^n \cdot \mathbf{u}_n$$

Predictor-corrector



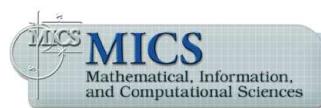
Example of Subcell Forces



Pressure force - $\int_{\partial V_n} p \mathbf{n} dS$

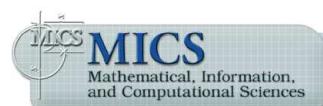
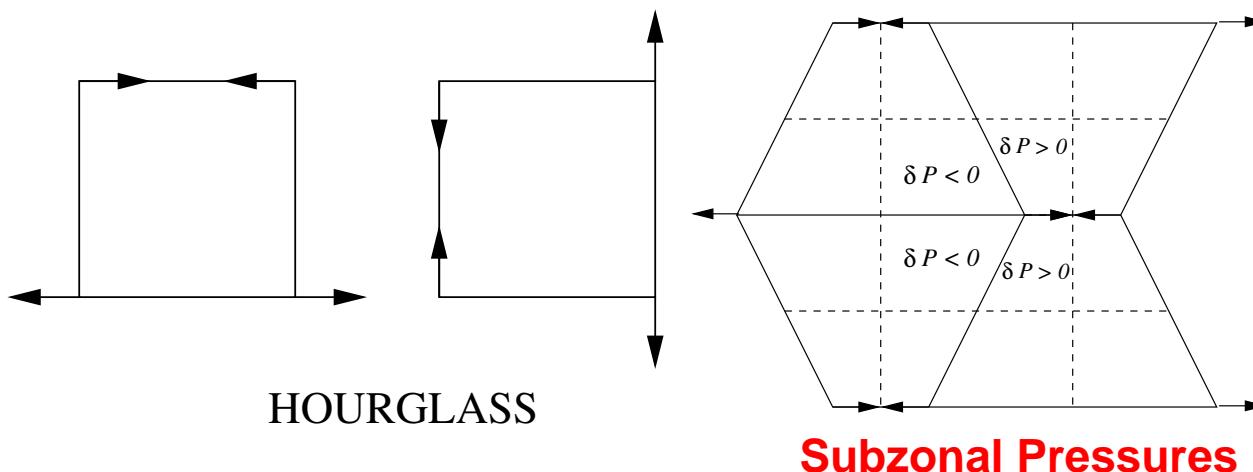
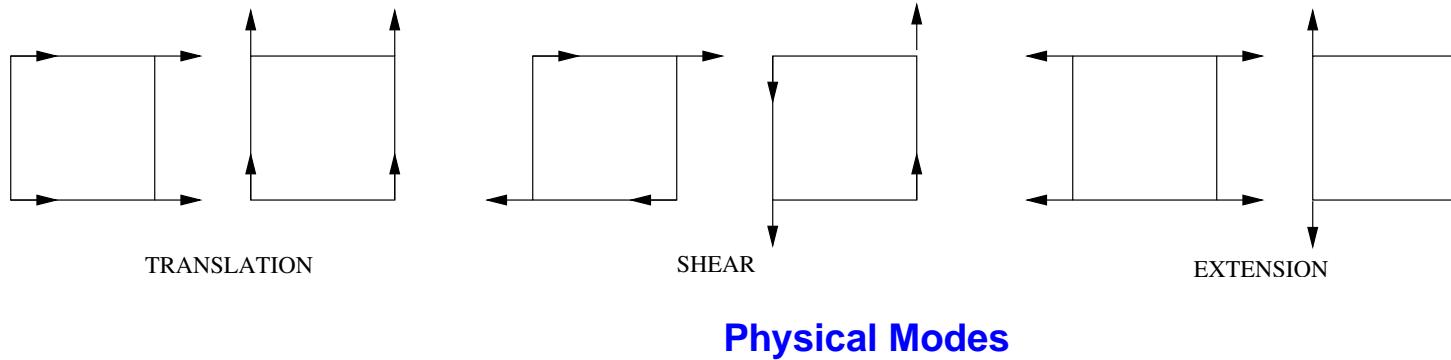
Subcell pressure force:

$$f_c^{n_1} \approx \int_{\{(n_1, n_2); c; (n_1, n_3)\}} p \mathbf{n} dS$$
$$p_{\{(n_1, n_2); c\}} = (p_{cn_1} + p_{cn_2})/2$$



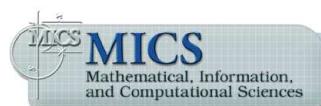
Hourglass - Parasitic Grid Motion

Reason - artificial null spaces of discrete divergence and gradient

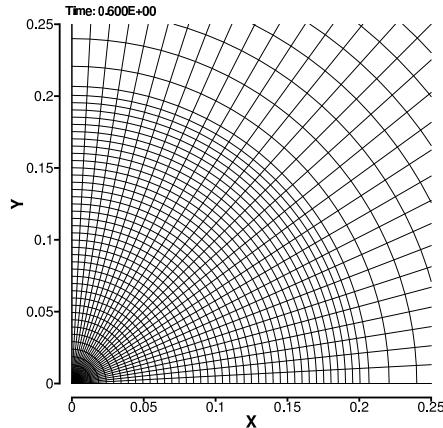


Mimetic Artificial Viscosity

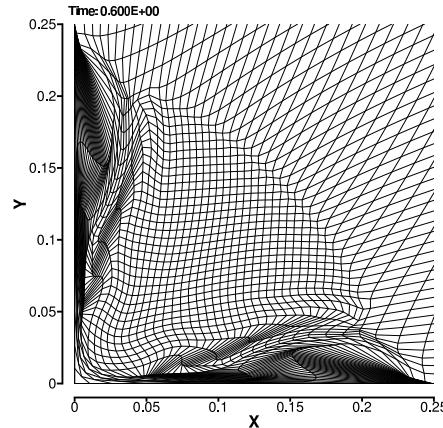
- **Dissipativity:** The artificial viscosity must only act to decrease kinetic energy;
- **Galilean invariance:** The viscosity should vanish smoothly as velocity field becomes constant;
- **Self-similar motion invariance:** The viscosity should vanish for uniform contraction or rigid rotation;
- **Wave-front invariance:** The viscosity should have no effect along a wave front constant phase on a grid aligned with the shock;
- **Viscous force continuity:** The viscous force should go to zero continuously as compression vanishes and remains zero for expansion.
- **In a limit (when grid is refined) artificial viscosity has to depend on flow and to be independent on the mesh.**



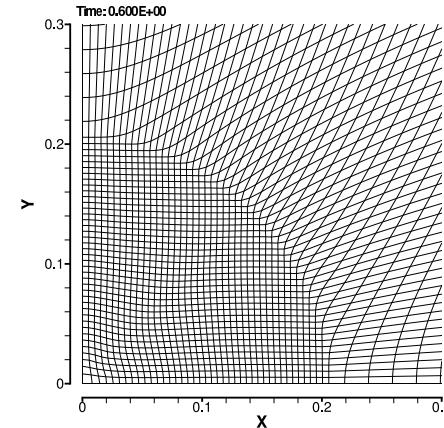
Mimetic Artificial Viscosity



Initial Polar Mesh
Edge Viscosity

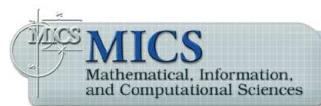


Initial Square Mesh
Edge Viscosity



Initial Square Mesh
Tensor Viscosity

The discrete viscosity tensor is formed on base of discretization of gradient of velocity tensor. Mimetic Discretizations are used to derive forms of the momentum and energy equations for general unstructured grids with the viscosity tensor evaluated at the cell edges



Rezone - Goal is to Improve Robustness and Accuracy

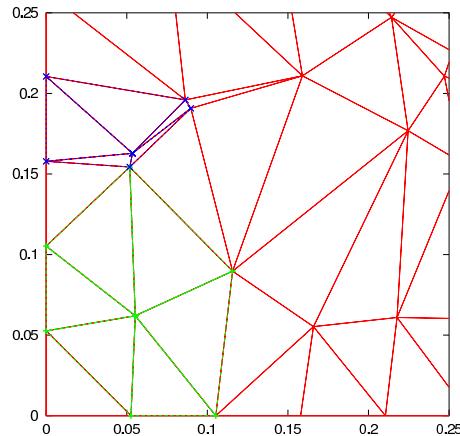
- As Close as Possible to Lagrangian
 - Preserve Information about Flow
 - Efficient (local) and Accurate Remap
- Improve Geometrical Quality
 - Accuracy of Lagrangian Phase

Reference Jacobian Rezone Strategy Optimization Framework

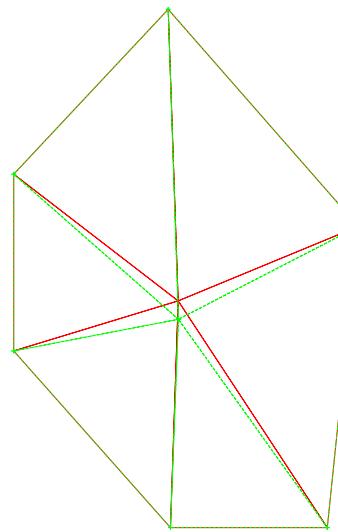
$$F(\{x_n\}) \sim \int_V \frac{\|J - J_{ref}\|}{\|J_{ref}\|} \frac{|J_{ref}|}{|J|} dV$$



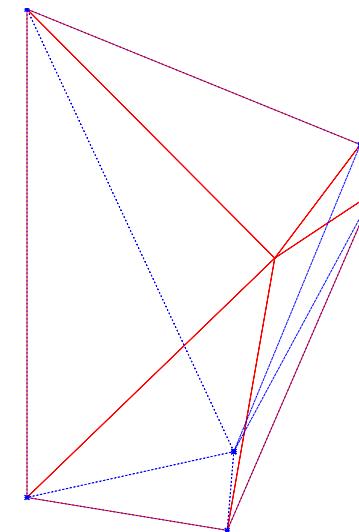
Construction of the RJM's from the Lagrangian Grid



a)



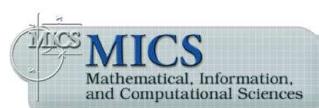
b)



c)

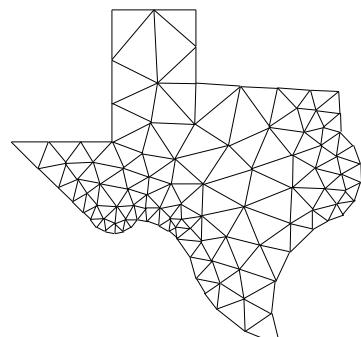
Local Smoothing on smooth and non-smooth patches of unstructured grid - fragment;

a) Patches on the grid, b) Smooth patch, c) Non-smooth patch.

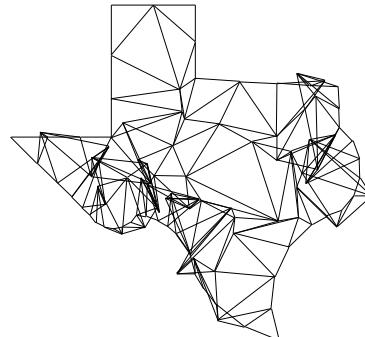


Mesh Untangling and Smoothing

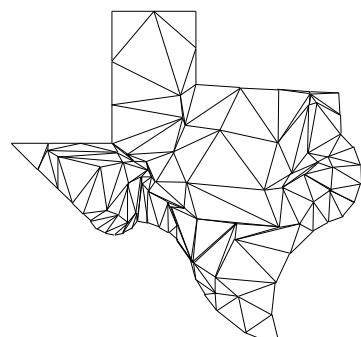
Optimization Algorithm for Reference Jacobian Rezone Strategy Requires Valid Mesh as Initial Guess → Mesh Untangling



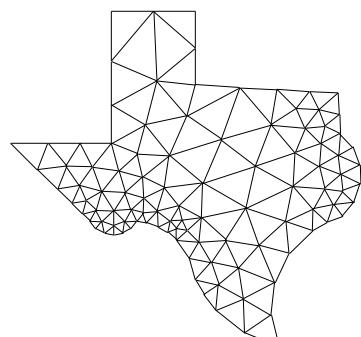
(a)



(b)



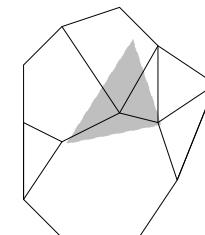
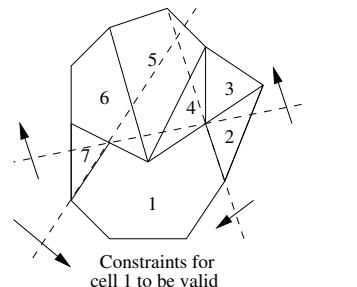
(c)



(d)

Three Stage Untangling
Algorithm:

Feasible Set

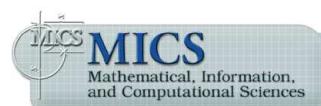


All constraints define
the feasible set

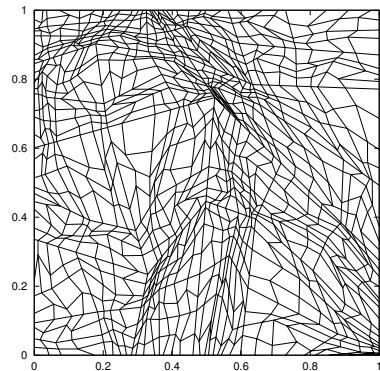
Optimization Based

$$\min_{x_n} \left(Vol^{\pm}(T_{cn}) - |Vol^{\pm}(T_{cn})| \right)^2$$

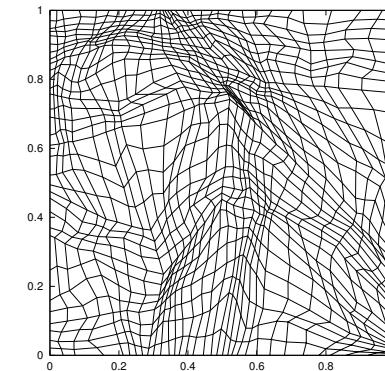
Feasible Set



Rezone Shestakov Mesh

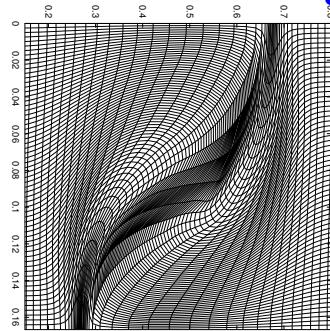


Original Mesh

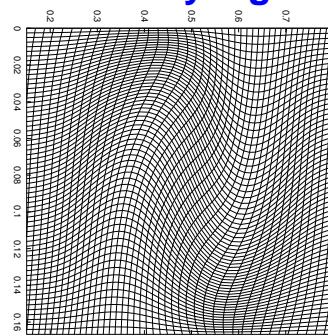


Mesh Improved by RJM

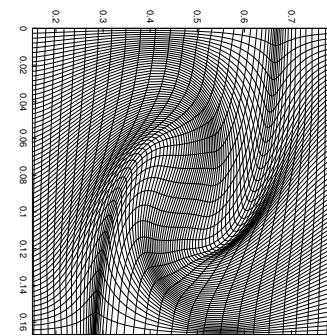
Mesh Fragments for Rayleigh-Taylor Problem



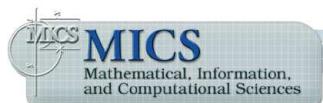
Lagrangian



Winslow



RJM



Staggered Remap - What is conserved in Lagrangian Phase

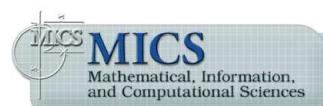
Cell Momentum - $\mu_c = \sum_{n \in N(c)} m_{cn} u_n, \nu_c = \sum_{n \in N(c)} m_{cn} v_n$

Cell Kinetic Energy - $K_c = \sum_{n \in N(c)} m_{cn} \frac{|\mathbf{u}_n|^2}{2}$

Cell Internal Energy - $\mathcal{E}_c = m_c \varepsilon_c$

Conserved Quantities Mass, Momentum, Total Energy

$$M = \sum_c m_c, \mu_u = \sum_c \mu_c, \mu_v = \sum_c \nu_c, E = \sum_c (\mathcal{E}_c + K_c),$$



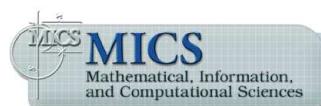
Staggered Remap - Statement

- **Given:**
 - Two grids: Lagrangian (old) - $\{c\}$ and Rezoned (new) - $\{\tilde{c}\}$
 - Density, internal energy and velocity on old mesh $\rho(cn), \varepsilon(c), u(n), v(n)$.
- **Find:**
 - Density, internal energy and velocity $\rho(\tilde{cn}), \varepsilon(\tilde{c}), u(\tilde{n}), v(\tilde{n})$ on the new mesh.



Staggered Remap - Requirements

- **Conservation:** $\tilde{M} = M$, $\tilde{\mu}_u = \mu_u$, $\tilde{\mu}_v = \mu_v$, $\tilde{E} = E$.
- **Reversibility:** If the new and old meshes are identical, then the remapped primary variables should show no change.
- **Bound-preservation:** The remapped density, velocity components and internal energy have to be contained within physically justified bounds.
- **Accuracy:**
 - Density - linearity-preserving.
 - Velocity - DeBar condition: exactness for a uniform velocity and spatially varying density.



Staggered Remap - Main Stages

- **Gathering stage**

Define mass, momentum, internal energy and kinetic energy in the subcells ensuring that the gathering stage is conservative.

- **Subcell remapping stage**

Conservative remapping of mass, momentum, internal and kinetic energy from subcells of Lagrangian mesh to subcells of rezoned mesh.

- **Scattering stage**

Conservatively recover the primary variables — subcell density, nodal velocity, and cell-centered specific internal energy — on the new rezoned mesh.



Gathering Stage

Design Principle - Conservation is Enforced on Cell Basis

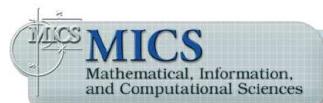
$$\sum_{n \in N(c)} \mu_{cn} = \mu_c, \quad \sum_{n \in N(c)} \mathcal{E}_{cn} = \mathcal{E}_c, \quad \sum_{n \in N(c)} K_{cn} = K_c,$$

Conservation

$$M^s \stackrel{def}{=} \sum_{cn} m(cn) = M ,$$

$$\mu_u^s \stackrel{def}{=} \sum_{cn} \mu(cn) = \mu_u , \quad \mu_v^s \stackrel{def}{=} \sum_{cn} \nu(cn) = \mu_v ,$$

$$\mathcal{E}^s \stackrel{def}{=} \sum_{cn} \mathcal{E}(cn) = \mathcal{E} , \quad K^s \stackrel{def}{=} \sum_{cn} K(cn) = K .$$



Gathering Stage - Definition of Subcell Momenta

$$\sum_{n \in N(c)} m(cn) u(cn) = \sum_{n \in N(c)} m(cn) u(n) = \mu(c).$$

$$u(cn) = \frac{u(c) + u(n) + u_{n,n+} + u_{n-,n}}{4}.$$

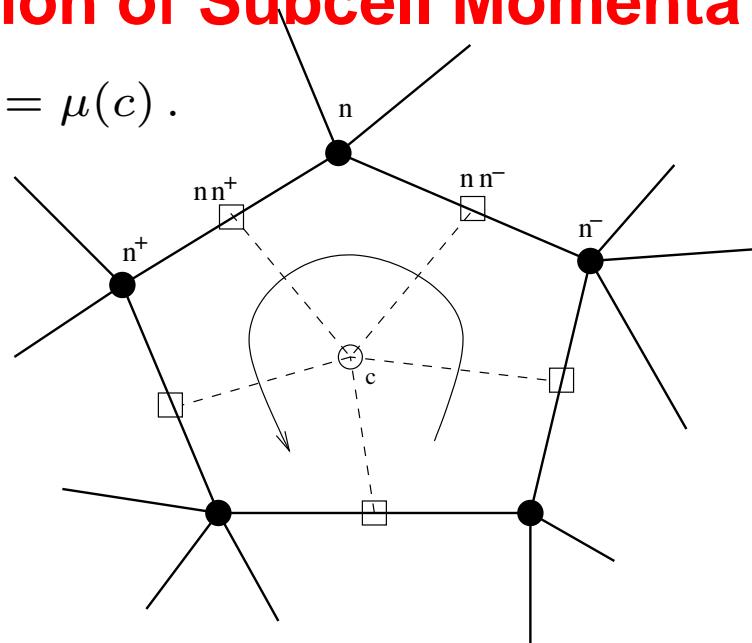
$$u_{n-,n} = \frac{1}{2} (u(n^-) + u(n)) ,$$

$$u_{n,n+} = \frac{1}{2} (u(n) + u(n^+)) ,$$

$u(c)$ — has to be defined

$$u(c) = \frac{1}{m(c)} \sum_{n \in N(c)} m(cn) u(n) - \frac{1}{m(c)} \sum_{n \in N(c)} m(cn) \frac{u(n^+) - 2u(n) + u(n^-)}{2},$$

$$\mathbf{1D} \rightarrow u_{i+\frac{1}{2}} = \left(m_{i+\frac{1}{2},i} u_i + m_{i+\frac{1}{2},i+1} u_{i+1} \right) / \left(m_{i+\frac{1}{2},i} + m_{i+\frac{1}{2},i+1} \right) ,$$

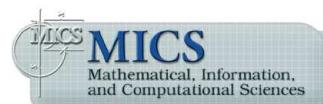


Gathering Stage - Definition of Subcell Momenta

$$\begin{aligned}
 u(cn) &= \frac{1}{4} \left(2 u_n + \frac{u_{n+}}{2} + \frac{u_{n-}}{2} \right) + \\
 &\quad \left[\sum_{k \in N(c)} u_k \left(\frac{-m_{ck-} + 4m_{ck} - m_{ck+}}{8m_c} \right) \right] . \\
 u_{i+\frac{1}{2},i} &= \frac{1}{2} u_i + \left[\frac{1}{2} \frac{1}{m_{i+\frac{1}{2}}} \left(m_{i+\frac{1}{2},i} u_i + m_{i+\frac{1}{2},i+1} u_{i+1} \right) \right] , \\
 u_{i+\frac{1}{2},i+1} &= \frac{1}{2} u_{i+1} + \left[\frac{1}{2} \frac{1}{m_{i+\frac{1}{2}}} \left(m_{i+\frac{1}{2},i} u_i + m_{i+\frac{1}{2},i+1} u_{i+1} \right) \right] .
 \end{aligned}$$

Matrix form

$$\mathbf{U}(c) = \{u(n), n \in N(c)\}^t, \mathbf{U}^s(c) = \{u(cn), n \in N(c)\}^t, \mathbf{U}^s(c) = \bar{\mathbf{I}}_c \mathbf{U}(c),$$



Gathering Stage - Definition of Subcell Momenta

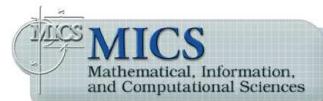
$$\bar{\mathbf{I}}_c = \frac{1}{4} \cdot \left(\begin{array}{cccccc} 2 & \frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & \frac{1}{2} & 2 \end{array} \right) + \left(\begin{array}{ccccc} Q_{N,1,2} & Q_{1,2,3} & Q_{2,3,4} & \cdots & Q_{N-1,N,1} \\ Q_{N,1,2} & Q_{1,2,3} & Q_{2,3,4} & \cdots & Q_{N-1,N,1} \\ Q_{N,1,2} & Q_{1,2,3} & Q_{2,3,4} & \cdots & Q_{N-1,N,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{N,1,2} & Q_{1,2,3} & Q_{2,3,4} & \cdots & Q_{N-1,N,1} \end{array} \right) .$$

$$Q_{n^-, n, n^+} = \frac{-m_{cn^-} + 4m_{cn} - m_{cn^+}}{8m_c} .$$

$$\bar{\mathbf{I}}_c = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2m_{i+\frac{1}{2}}} \begin{pmatrix} m_{i+\frac{1}{2}, i} & m_{i+\frac{1}{2}, i+1} \\ m_{i+\frac{1}{2}, i} & m_{i+\frac{1}{2}, i+1} \end{pmatrix}$$

Matrix $\bar{\mathbf{I}}_c$ easily invertible for arbitrary number of edges of polygon and transform constant velocity to constant velocity

$$\bar{\mathbf{I}}_c \mathbf{C} = \mathbf{C} , \quad (\bar{\mathbf{I}}_c)^{-1} \mathbf{C} = \mathbf{C} , \quad \mathbf{C} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_N)$$



Gathering Stage - Definition of Subcell Kinetic Energy

$$\mathbf{k}(c) = \left\{ k(n) = \frac{|\mathbf{u}(n)|^2}{2}, \quad n \in N(c) \right\}^t, \quad \mathbf{k}^s(c) = \{k(cn), \quad n \in N(c)\}^t,$$

By definition

$$\mathbf{k}^s(c) \stackrel{def}{=} \mathbf{l}_c \mathbf{k}(c).$$

Subcell Kinetic Energy

$$K(cn) = m(cn)k(cn).$$

We emphasize that

$$k(cn) \neq \frac{|\mathbf{u}(cn)|^2}{2}.$$



Gathering Stage - Definition of Subcell Internal Energy

- **Conservative piece-wise linear reconstruction**

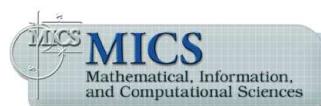
$$(\rho\varepsilon)_c(x, y) = \rho(c)\varepsilon(c) + \delta_x^c(x - x_c) + \delta_y^c(y - y_c) ,$$

$$\int_c (\rho\varepsilon)_c(x, y) dx dy = \mathcal{E}(c) = \rho(c) \varepsilon(c) V(c) = m(c) \varepsilon(c) .$$

Slopes δ_x^c, δ_y^c are defined by Barth-Jespersen algorithm.

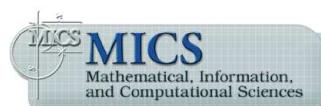
- **$\mathcal{E}(cn)$ is obtained by integration of $(\rho\varepsilon)_c(x, y)$ over the subcell**

$$\mathcal{E}(cn) = \int_{cn} (\rho\varepsilon)_c(x, y) dx dy.$$

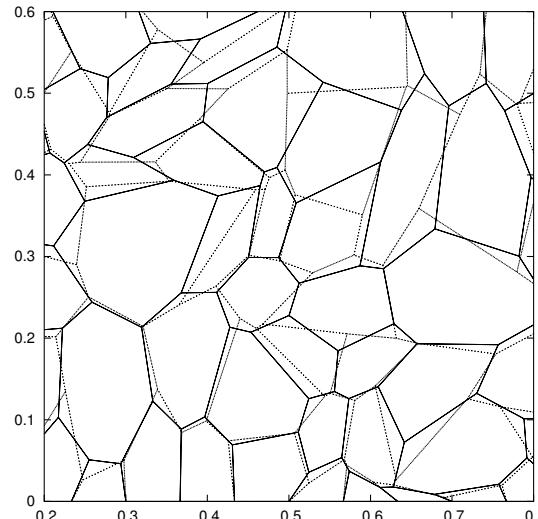


Subcell Remapping Stage

- **Given:**
 - Two grids: Lagrangian and Rezoned (here cells are subcells)
 - Mean values of underlying functions: density (momentum, total energy) in the Lagrangian cells
- **Find mean values of density in Rezoned cells**
 - Accuracy, Linearity Preservation
 - Bound-Preservation (for example, positivity of density)
 - Conservation - total mass is conserved



Remap - Goal is Accurate Conservative Interpolation



Lagrangian and rezoned grid

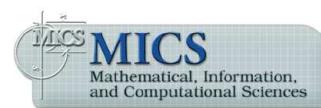
$$\widetilde{M}^s \stackrel{\text{def}}{=} \sum_{\widetilde{cn}} m(\widetilde{cn}) = M^s ,$$

$$\widetilde{\mu}_u^s \stackrel{\text{def}}{=} \sum_{\widetilde{cn}} \mu(\widetilde{cn}) = \mu_u^s ,$$

$$\widetilde{\mu}_v^s \stackrel{\text{def}}{=} \sum_{\widetilde{cn}} \nu(\widetilde{cn}) = \mu_v^s ,$$

$$\widetilde{\mathcal{E}}^s \stackrel{\text{def}}{=} \sum_{\widetilde{cn}} \mathcal{E}(\widetilde{cn}) = \mathcal{E}^s ,$$

$$\widetilde{K}^s \stackrel{\text{def}}{=} \sum_{\widetilde{cn}} K(\widetilde{cn}) = K^s$$



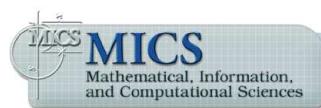
Remap - Principles

- **Conservative piecewise linear reconstruction on old Lagrangian mesh, Limiters**
- **Swept region integration over cells of rezoned mesh - approximate quadrature, which does not require cell intersections — Efficiency**
- **Repair - No new extremes are created**



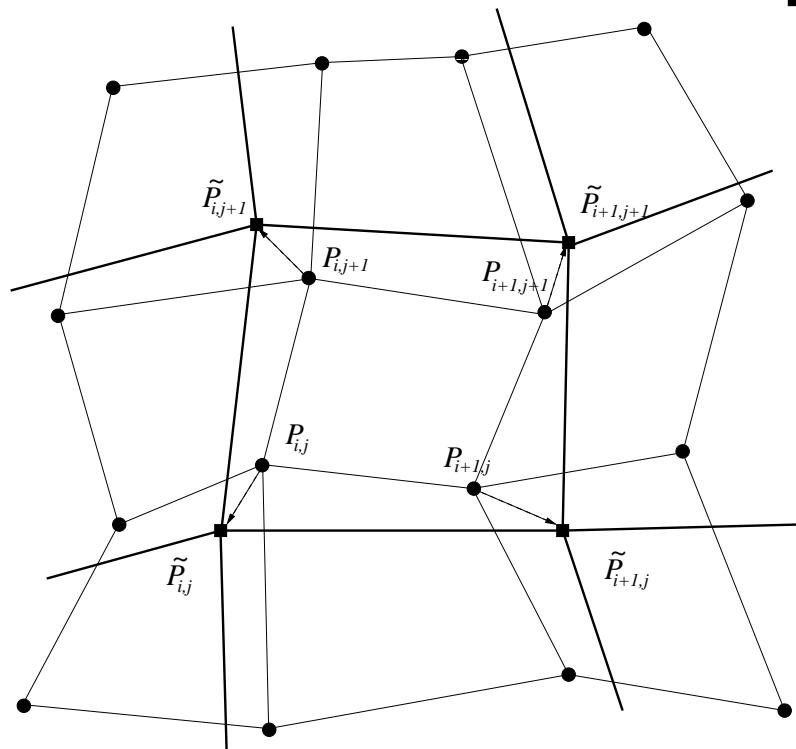
Numerical Quadrature — Main Ideas

- Smooth function can be approximated by polynomial function (truncated Taylor series)
- Volume integral of polynomial function can be reduced to boundary integral
- Boundary integral over new cell can be decomposed in sum of integrals over swept regions (regions which are covered by the continuous movement of the faces from their old to their new positions)
- Boundary integral over polygon can be computed exactly



Numerical Quadrature

Decomposition of the boundary of the new mesh



$$\begin{aligned}\{\tilde{P}_{i,j}, \tilde{P}_{i+1,j}, \tilde{P}_{i+1,j+1}, \tilde{P}_{i,j+1}\} = \\ \{P_{i,j}, P_{i+1,j}, P_{i+1,j+1}, P_{i,j+1}\} + \\ + \{P_{i+1,j}, \tilde{P}_{i+1,j}, \tilde{P}_{i+1,j+1}, P_{i+1,j+1}\} \\ - \{P_{i,j}, \tilde{P}_{i,j}, \tilde{P}_{i,j+1}, P_{i,j+1}\} \\ + \{P_{i,j+1}, P_{i+1,j+1}, \tilde{P}_{i+1,j+1}, \tilde{P}_{i,j+1}\} \\ - \{P_{i,j}, P_{i+1,j}, \tilde{P}_{i+1,j}, \tilde{P}_{i,j}\}\end{aligned}$$

Swept Regions, Line Integrals and Exact Mass

Regions which are covered by the continuous movement of the faces from their old to their new positions.

$$\delta F_{i+\frac{1}{2},j} = \{P_{i,j}, P_{i+1,j}, \tilde{P}_{i+1,j}, \tilde{P}_{i,j}\}$$

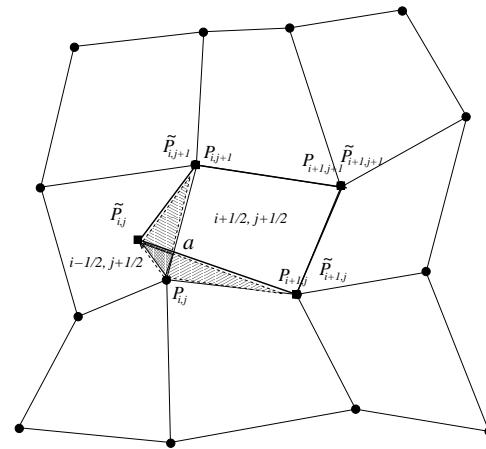
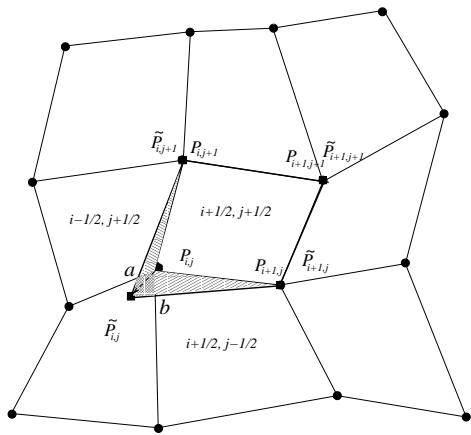
Exact Mass and Line Integrals over Swept Regions

$$\tilde{m}_{i+\frac{1}{2},j+\frac{1}{2}}^{ex} = m_{i+\frac{1}{2},j+\frac{1}{2}} + \mathcal{F}_{i+1,j+\frac{1}{2}}^{ex} - \mathcal{F}_{i,j+\frac{1}{2}}^{ex} + \mathcal{F}_{i+\frac{1}{2},j+1}^{ex} - \mathcal{F}_{i+\frac{1}{2},j}^{ex},$$

$$\mathcal{F}_{i,j+\frac{1}{2}}^{ex} = \oint_{\partial(\delta F_{i,j+\frac{1}{2}})} \left(g x + \frac{\partial g}{\partial x} \cdot \frac{(x - x_{i,j+\frac{1}{2}})^2}{2} + \frac{\partial g}{\partial y} \cdot x (y - y_{i,j+\frac{1}{2}}) \right)$$



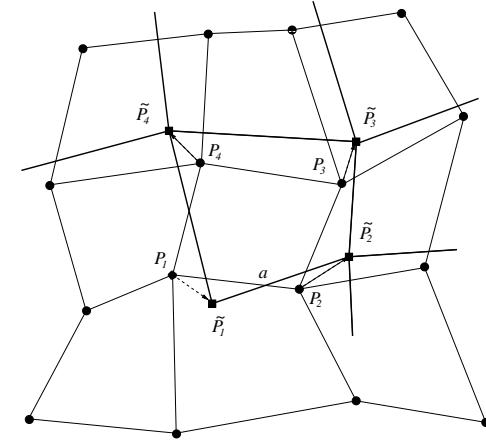
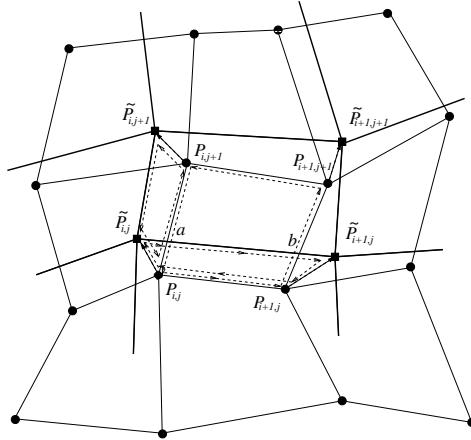
Swept Regions and Signed Volumes



Swept region — $\delta F_{i+\frac{1}{2},j} = \{P_{i,j}, P_{i+1,j}, \tilde{P}_{i+1,j}, \tilde{P}_{i,j}\}$,

Signed volume — $V(\delta F_{i+\frac{1}{2},j}) = \oint_{P_{i,j}, P_{i+1,j}, \tilde{P}_{i+1,j}, \tilde{P}_{i,j}} x dy$

Face-Based Remapping



$$\tilde{m}_{i+\frac{1}{2},j+\frac{1}{2}}^* = m_{i+\frac{1}{2},j+\frac{1}{2}} + \mathcal{F}_{i+1,j+\frac{1}{2}}^* - \mathcal{F}_{i,j+\frac{1}{2}}^* + \mathcal{F}_{i+\frac{1}{2},j+1}^* - \mathcal{F}_{i+\frac{1}{2},j}^*,$$

$$\mathcal{F}_{i,j+\frac{1}{2}}^* = \int_{\delta F_{i,j+\frac{1}{2}}} g_{i,j+\frac{1}{2}}(x, y) \, dV$$

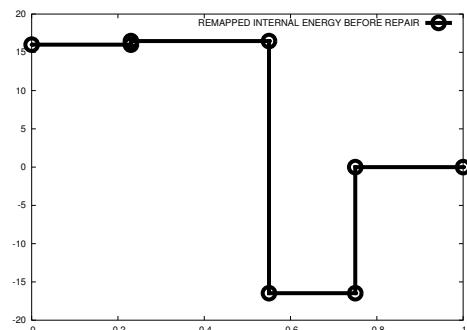
$$g_{i,j+\frac{1}{2}}(x, y) = \begin{cases} g_{i+\frac{1}{2},j+\frac{1}{2}}(x, y), & V(\delta F_{i,j+\frac{1}{2}}) \geq 0, \\ g_{i-\frac{1}{2},j+\frac{1}{2}}(x, y), & V(\delta F_{i,j+\frac{1}{2}}) < 0. \end{cases}$$



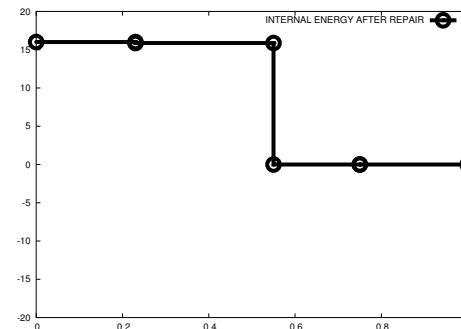
Repair - Conservative Redistribution

Goal - to enforce local bounds (in particular positivity of density and internal energy)

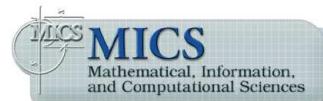
Strategy - redistribution of mass, momentum and total energy among the cells of the mesh.



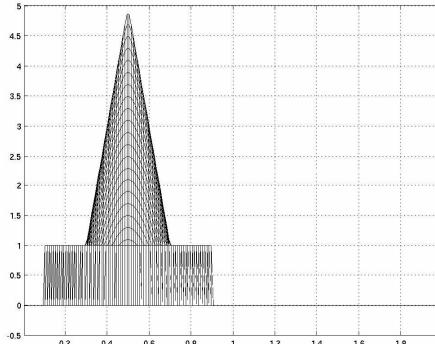
**Internal Energy After Remap and
Before Repair**



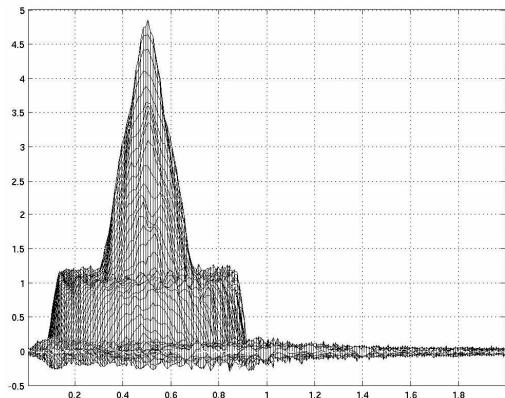
Internal Energy After Repair



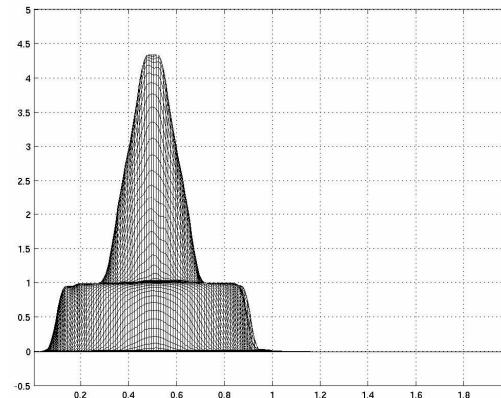
Repair - Advection of Cone - 3 rotation



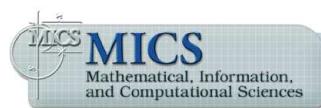
Initial



Before Repair



After Repair



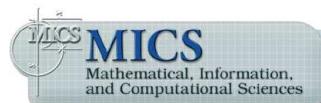
Scattering Conservation Requirements

$$\widetilde{M} = \sum_{\widetilde{cn}} m_{\widetilde{cn}} = \sum_{\widetilde{cn}} \rho_{\widetilde{cn}} V_{\widetilde{cn}} = \widetilde{M}^s,$$

$$\widetilde{\mu}_u = \sum_{\widetilde{n}} m_{\widetilde{n}} u_{\widetilde{n}} = \widetilde{\mu}_u^s, \quad \widetilde{\mu}_v = \sum_{\widetilde{n}} m_{\widetilde{n}} v_{\widetilde{n}} = \widetilde{\mu}_v^s,$$

$$\begin{aligned} \widetilde{E} &= \sum_{\widetilde{c}} (\mathcal{E}(\widetilde{c}) + K(\widetilde{c})) = \sum_{\widetilde{c}} \left[m_{\widetilde{c}} \varepsilon_{\widetilde{c}} + \sum_{\widetilde{n} \in N(\widetilde{c})} m_{\widetilde{cn}} \frac{|\mathbf{u}_{\widetilde{n}}|^2}{2} \right] \\ &= \widetilde{\mathcal{E}}^s + \widetilde{K}^s = \widetilde{E}^s \end{aligned}$$

The Subcell Density - $\rho_{\widetilde{cn}} = m_{\widetilde{cn}} / V_{\widetilde{cn}}$.



Scattering - Definition of Nodal Velocity

Subcell velocity

$$u_{\tilde{c}\tilde{n}} \stackrel{\text{def}}{=} \frac{\mu_{\tilde{c}\tilde{n}}}{m_{\tilde{c}\tilde{n}}}.$$

One-sided nodal velocities

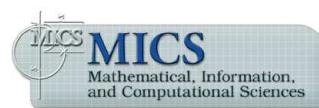
$$\mathbf{U}_{\tilde{c}} = (\bar{\mathbf{I}}_{\tilde{c}})^{-1} \mathbf{U}^s_{\tilde{c}}, \quad \mathbf{U}_{\tilde{c}} = \left\{ u_{\tilde{n}}^{\tilde{c}}, \quad \tilde{n} \in N(\tilde{c}) \right\}^t, \quad \mathbf{U}^s_{\tilde{c}} = \{ u_{\tilde{c}\tilde{n}}, \quad \tilde{n} \in N(\tilde{c}) \}^t.$$

Unique nodal velocity

$$u_{\tilde{n}} = \frac{1}{m_{\tilde{n}}} \sum_{\tilde{c} \in C(\tilde{n})} m_{\tilde{c}\tilde{n}} u_{\tilde{n}}^{\tilde{c}}.$$

Final Kinetic Energy

$$K_{\tilde{c}} = \sum_{\tilde{n} \in N(\tilde{c})} m_{\tilde{c}\tilde{n}} \frac{|\mathbf{u}_{\tilde{n}}|^2}{2}.$$



Scattering - Definition of Cell-Centered Internal Energy

By definition

$$E_{\tilde{c}} = \mathcal{E}_{\tilde{c}} + K_{\tilde{c}},$$

$K_{\tilde{c}}$ — is known, $\mathcal{E}_{\tilde{c}}$ — has to be defined.

We know total energy

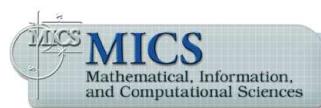
$$E_{\tilde{c}} = \sum_{\tilde{n} \in N(\tilde{c})} (\mathcal{E}_{\tilde{c}\tilde{n}} + K_{\tilde{c}\tilde{n}}) .$$

To be conservative internal energy has to be defined as follows

$$\mathcal{E}_{\tilde{c}} \stackrel{\text{def}}{=} \sum_{\tilde{n} \in N(\tilde{c})} \mathcal{E}_{\tilde{c}\tilde{n}} + \left[\left(\sum_{\tilde{n} \in N(\tilde{c})} K_{\tilde{c}\tilde{n}} \right) - K_{\tilde{c}} \right] .$$

Specific internal energy

$$\varepsilon_{\tilde{c}} = \mathcal{E}_{\tilde{c}} / m_{\tilde{c}} .$$



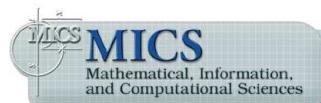
Properties of the Algorithm

- **Conservation**

Mass, momentum, and total energy are all conserved at each stage: gathering, subcell remapping, and scattering:

$$\begin{aligned}\widetilde{M} &= \widetilde{M}^s = M^s = M, \\ \widetilde{\mu}_u &= \widetilde{\mu}_u^s = \mu_u^s = \mu_u, \\ \widetilde{\mu}_v &= \widetilde{\mu}_v^s = \mu_v^s = \mu_v, \\ \widetilde{E} &= \widetilde{E}^s = E^s = E.\end{aligned}$$

That is, mass, momenta and total energy are conserved by the overall process.



Properties of the Algorithm

- **Reversibility**

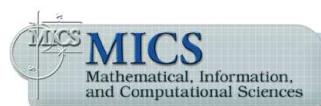
- Subcell remapping stage - no change in the subcell quantities
- Scattering Stage - $\bar{\mathbf{l}}_{\tilde{c}} = \bar{\mathbf{l}}_c \rightarrow u^{\tilde{c}}(\tilde{n}) = u(n)$

$$u(\tilde{n}) = \frac{1}{m(\tilde{n})} \sum_{\tilde{c} \in C(\tilde{n})} m(\tilde{c}\tilde{n}) u^{\tilde{c}}(\tilde{n}) =$$
$$\frac{1}{m(n)} \sum_{c \in C(n)} m(cn) u(n) = u(n) \left(\frac{1}{m(n)} \sum_{c \in C(n)} m(cn) \right) = u(n),$$

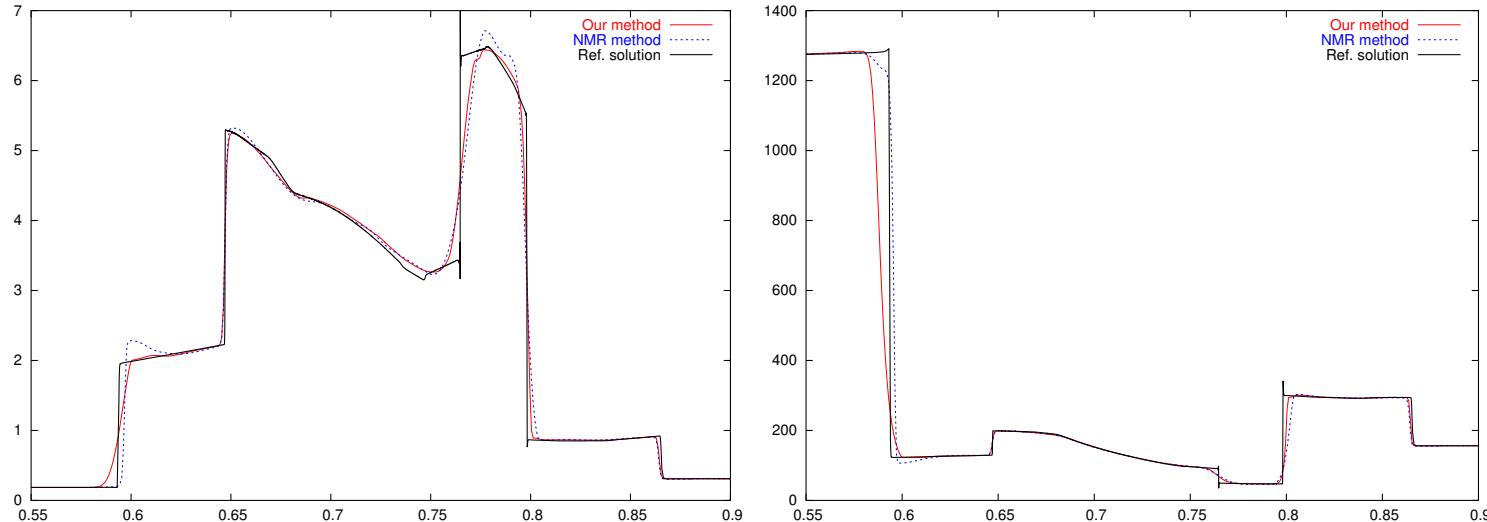
Kinetic energy does not change → Internal energy does not change

- **Accuracy**

- Linearity preservation on subcell remapping stage
- Bound preservation - repair
- DeBar Condition for velocity
- Numerical Examples



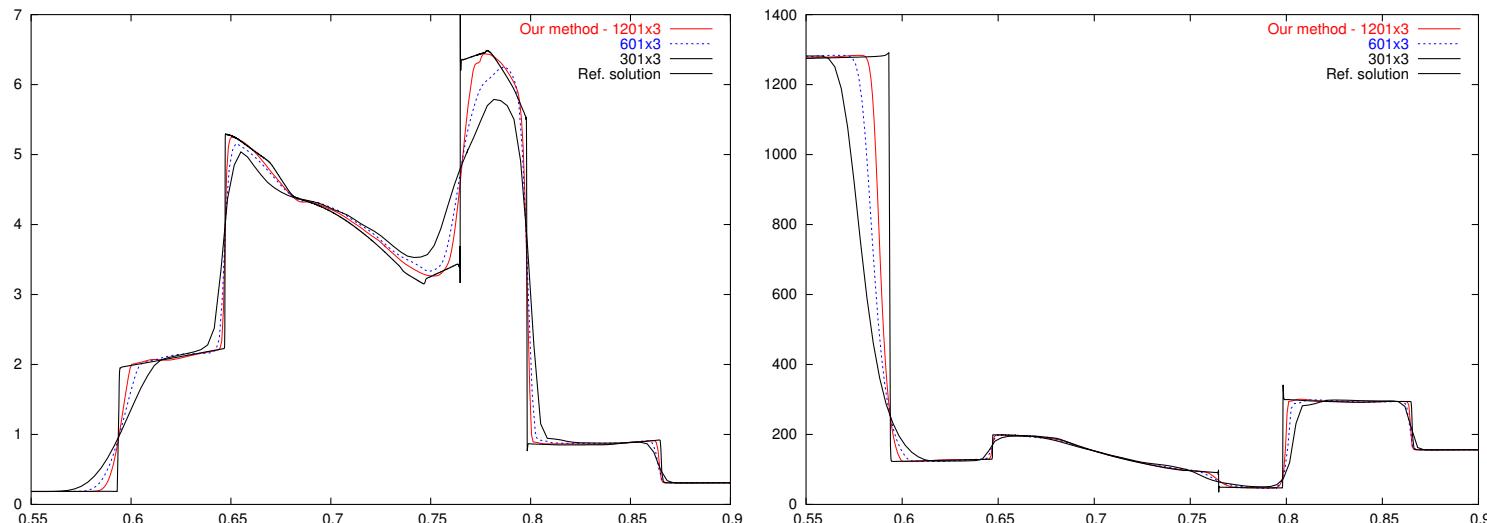
Numerical Examples - 1D Woodward-Colella Blast Wave Problem - Eulerian Frame



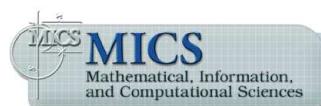
Woodward-Colella Blast Wave Problem. Comparison of the NMR method (Pember, Anderson - LLNL) and the new method: density (zoom) — left, and specific internal energy (zoom) — right at $t = 0.038$, $N_x = 1200$.



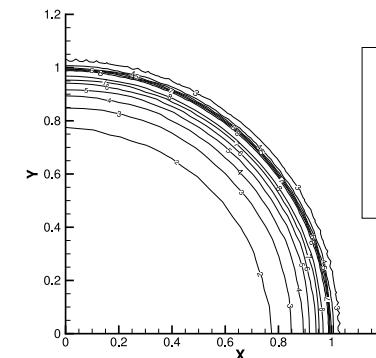
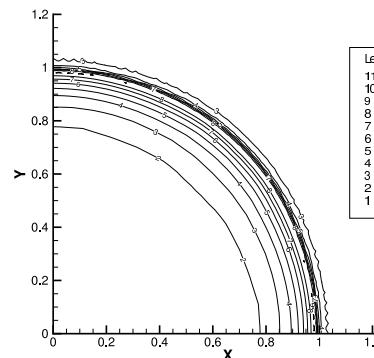
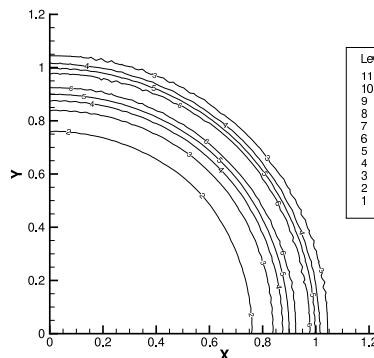
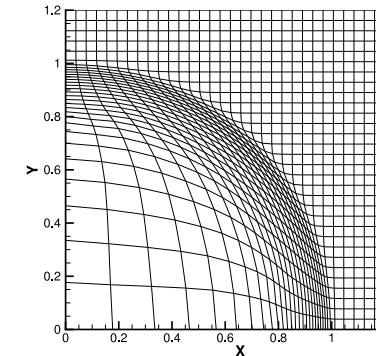
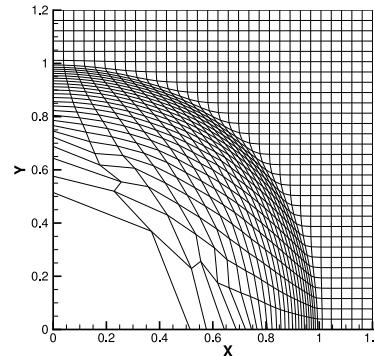
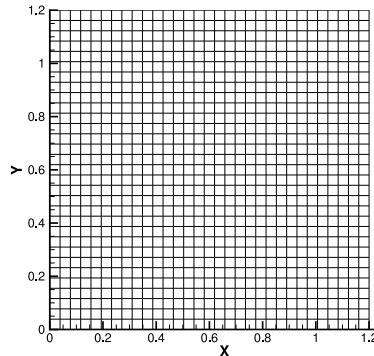
Numerical Examples - 1D Woodward-Colella Blast Wave Problem - Eulerian Frame



Convergence for Woodward-Colella Blast Wave Problem. Density (zoom) — left, and specific internal energy (zoom) — right at $t = 0.038$, $N_x = 300, 600, 1200$.



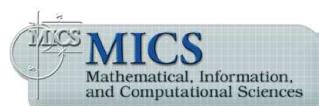
Numerical Examples - 2D Sedov Problem



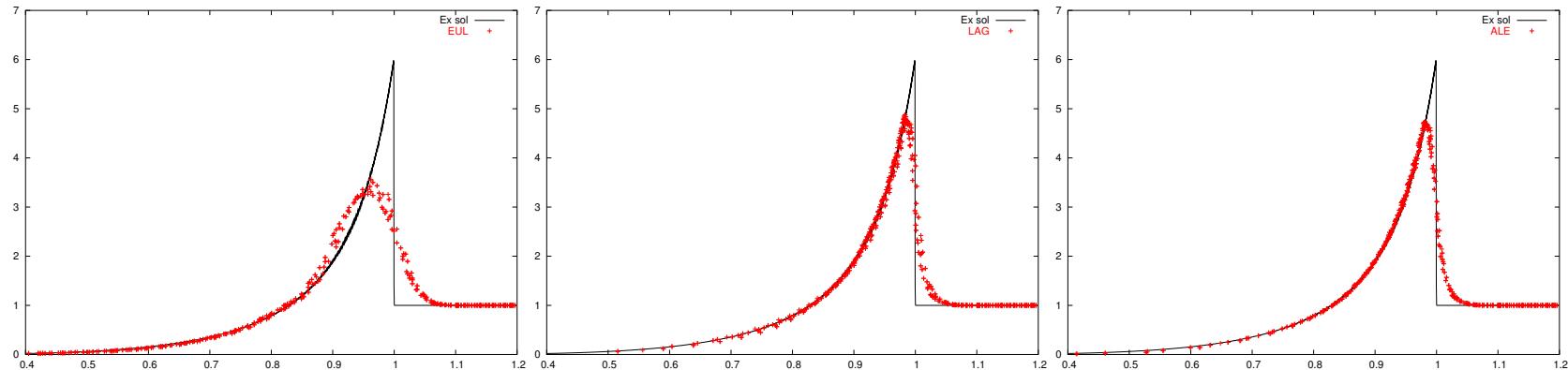
Eulerian

Lagrangian

ALE

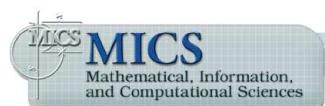


Numerical Examples - 2D Sedov Problem

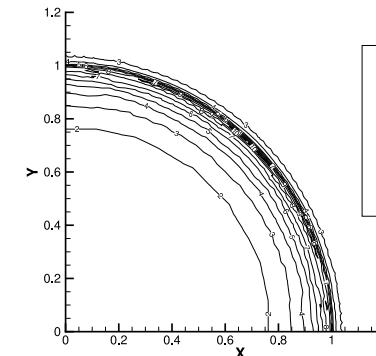
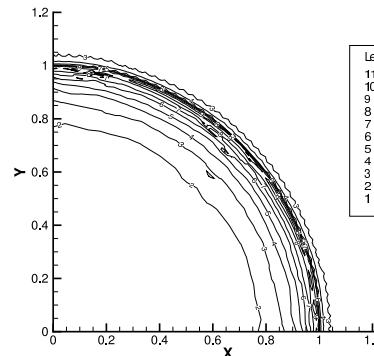
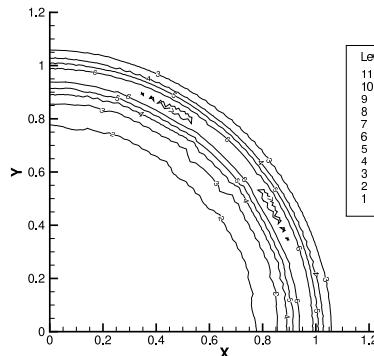
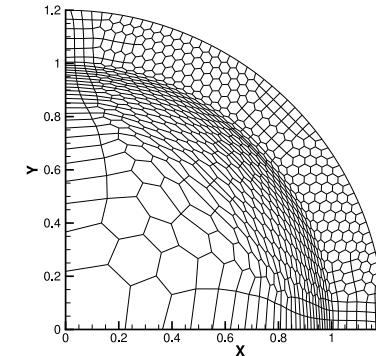
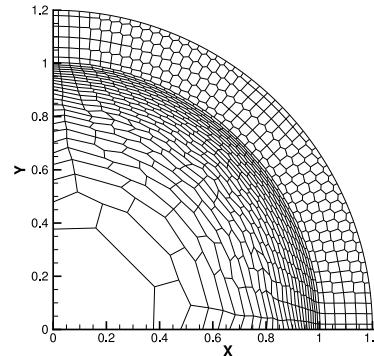
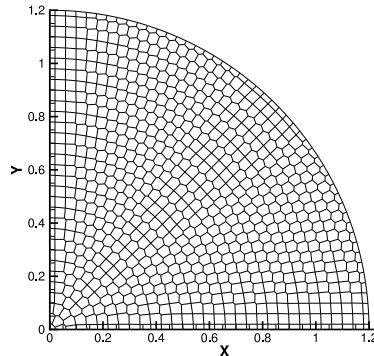


Density at $t = 1.0$ as a function of the radius (solid line exact solution) —
Eulerian regime (left), Lagrangian regime (middle), ALE regime (right).

	# of time steps	Peak density	Relative timing
Eulerian	477	3.55	10
Lagrangian	375	4.90	1
ALE-10	338	4.75	2



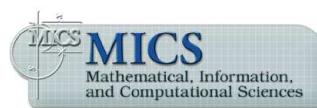
Numerical Examples - 2D Sedov Problem



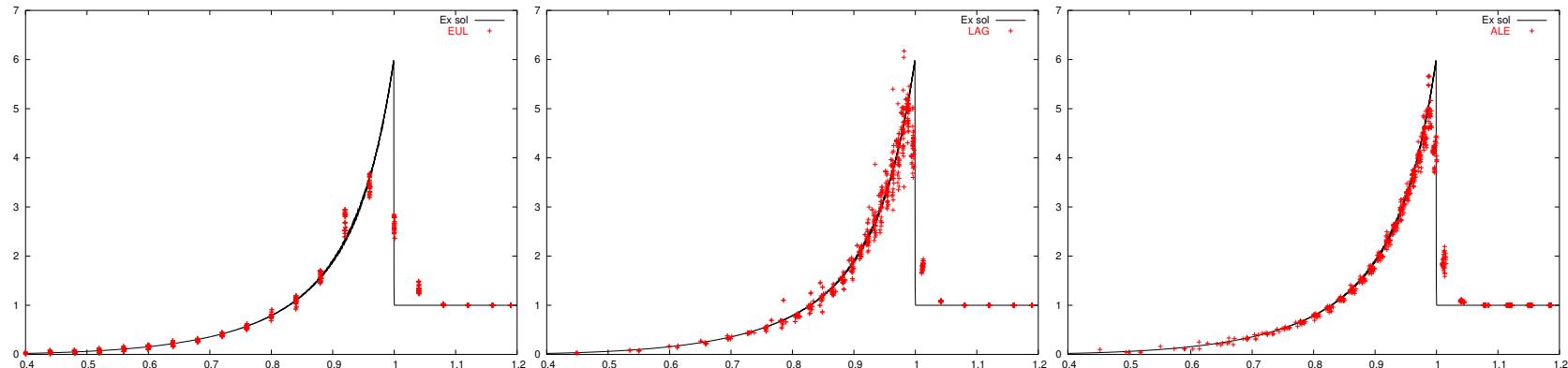
Eulerian

Lagrangian

ALE

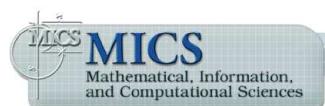


Numerical Examples - 2D Sedov Problem

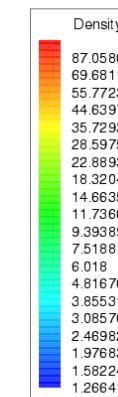
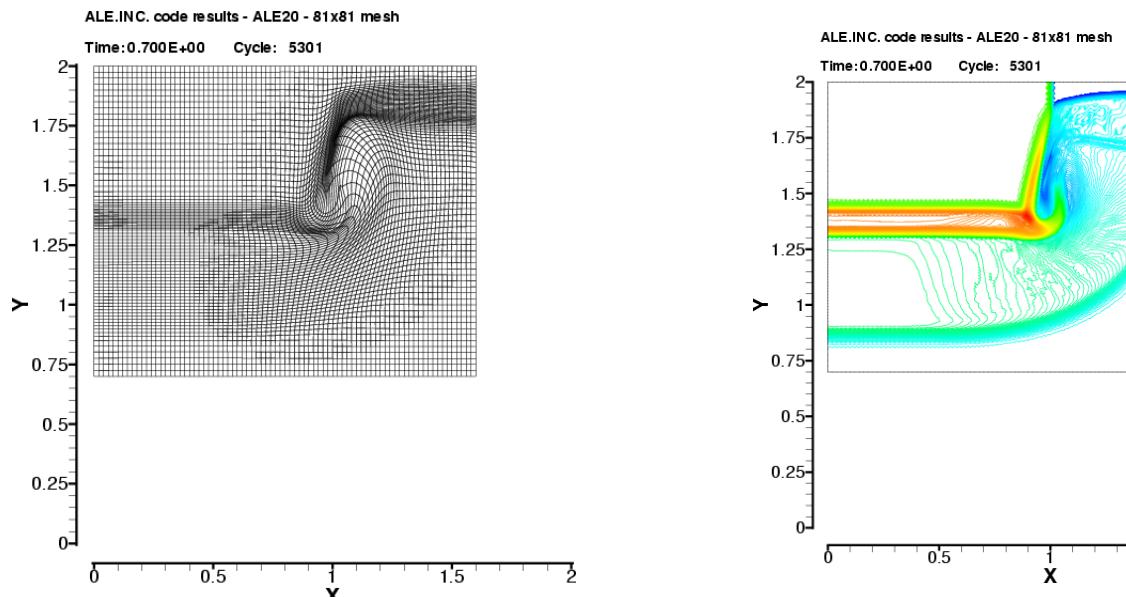


Density at $t = 1.0$ as a function of the radius (solid line exact solution) —
Eulerian regime (left), Lagrangian regime (middle), ALE regime (right).

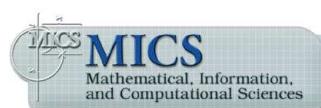
	# of time steps	Peak density	Relative timing
Eulerian	1567	3.69	20
Lagrangian	603	6.20	1
ALE-10	408	5.70	2



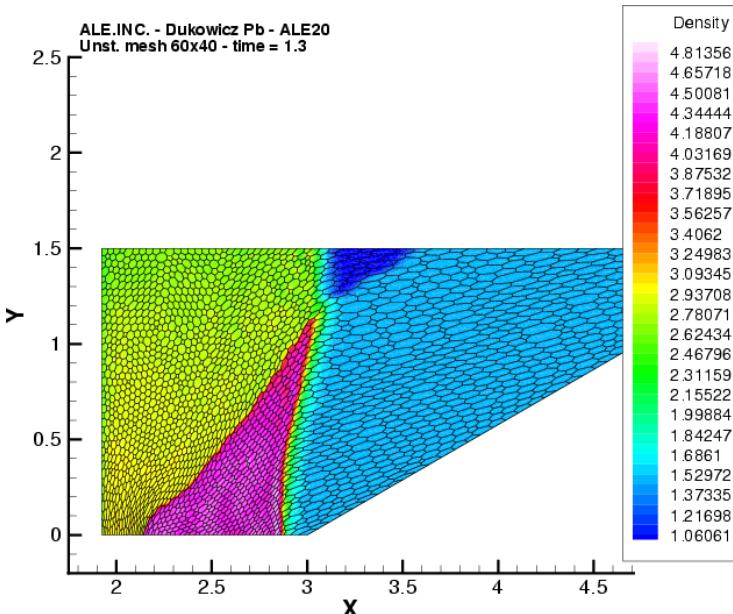
Examples of ALE Calculations



Interaction of shock with heavy obstacle

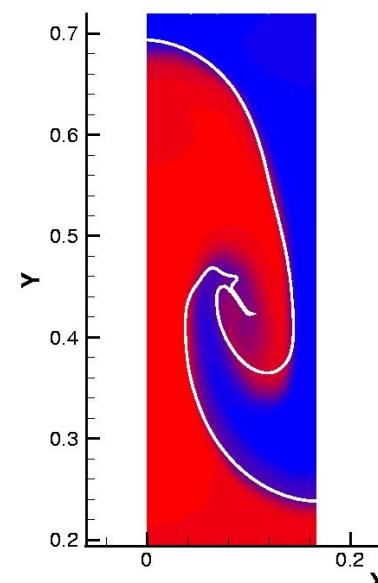


Examples of ALE Calculations



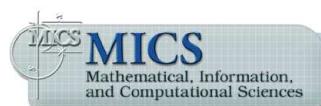
Shock Refraction Problem

General Polygonal Mesh



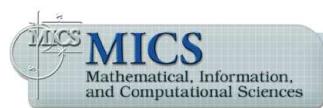
Rayleigh-Taylor Instability

Eulerian=Lagrange+Remap



References

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- E. J. Caramana and M. Shashkov, *Elimination of Artificial Grid Distortion and Hourglass-Type Motions by Means of Lagrangian Subzonal Masses and Pressures*, JCP, 142, pp. 521–561, (1998).
- J. Campbell and M. Shashkov, *A Tensor Artificial Viscosity Using a Mimetic Finite Difference Algorithm*, JCP, 172 (2001), pp.739–765.
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